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## ON UNIFORM DISTRIBUTION FOR INVARIANT EXTENSIONS OF THE LINEAR LEBESGUE MEASURE

### Abstract

The concept of uniform distribution in  $[0, 1]$  is extended for a certain strictly separated maximal (in the sense of cardinality) family  $(\lambda_t)_{t \in [0, 1]}$  of invariant extensions of the linear Lebesgue measure  $\lambda$  in  $[0, 1]$ , and it is shown that the  $\lambda_t^\infty$  measure of the set of all  $\lambda_t$ -uniformly distributed sequences is equal to 1, where  $\lambda_t^\infty$  denotes the infinite power of the measure  $\lambda_t$ . This is an analogy of Hlawka's (1956) theorem for  $\lambda_t$ -uniformly distributed sequences. An analogy of Weyl's (1916) theorem is obtained in the similar manner.

### 1 Introduction

The theory of uniform distribution is concerned with the distribution of real numbers in the unit interval  $(0, 1)$  and its development started with Hermann Weyl's celebrated paper [22]. This theory gives a useful technique for numerical calculation exactly of the one-dimensional Riemann integral over  $[0, 1]$ .

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More precisely, the sequence of real numbers  $(x_n)_{n \in \mathbb{N}} \in [0, 1]^\infty$  is uniformly distributed in  $[0, 1]$  if and only if for every real-valued Riemann integrable function  $f$  on  $[0, 1]$  the equality

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N f(x_n)}{N} = \int_0^1 f(x) dx. \quad (1.1)$$

holds( see, for example [11], Corollary 1.1, p. 3). Main corollaries of this assertion successfully were used in Diophantine approximations and have applications to Monte-Carlo integration (cf. [22],[24],[23],[11]). Note that the set  $S$  of all uniformly distributed sequences in  $[0, 1]$  viewed as a subset of  $[0, 1]^\infty$  has full  $\lambda^\infty$ -measure, where  $\lambda^\infty$  denotes the infinite power of the linear Lebesgue measure  $\lambda$  in  $[0, 1]$ ( cf. [11], Theorem 2.2(Hlawka), p. 183). For a fixed Lebesgue integrable function  $f$  in  $[0, 1]$ , one can state a question asking *what is a maximal subset  $S_f$  of  $S$  each element of which can be used for calculation it's Lebesgue integral over  $[0, 1]$  by the formula (1.1) and whether this subset has the full  $\lambda^\infty$ -measure.* This question has been resolved positively by Kolmogorov's Strong Law of Large Numbers. There naturally arises another question asking whether can be developed analogous methodology for invariant extensions of the Lebesgue measure in  $[0, 1]$  and whether main results of the uniform distribution theory will be preserved in such a situation. In the present manuscript we consider this question for a certain strictly separated<sup>1</sup> maximal (in the sense of cardinality) family of invariant extensions of the linear Lebesgue measure in  $[0, 1]$ . In our investigations we essentially use the methodology developed in works [5], [19], [11].

The rest of the present paper it the following.

In Section 2 we consider some auxiliary facts from the theory of invariant extensions of the Lebesgue measure and from the probability theory. In Section 3 we present our main results. Section 4 presents historical background of the theory of invariant extensions of the  $n$ -dimensional Lebesgue measure as well Haar measure in locally compact Hausdorff topological groups. In Section 5 we state a uniform distribution problem for invariant extensions of the Haar measure in locally compact Hausdorff topological groups.

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<sup>1</sup>The family of probability measures  $(\mu_i)_{i \in I}$  defined on the measure space  $(E, S)$  is called strictly separated if there exists a partition  $\{E_i : i \in I \text{ \& } E_i \in S\}$  of the set  $E$  such that  $\mu_i(E_i) = 1$  for  $i \in I$ .

## 2 Some auxiliary notions and facts from the theory of invariant extensions of the Lebesgue measure

As usual, we denote by  $R$  the real axis with usual metric and addition "+" operation under which  $R$  stands a locally compact  $\sigma$ -compact Hausdorff topological group. We denote by  $\lambda$  the linear Lebesgue measure in  $R$ .

**Lemma 2.1.** ([5], Lemma 6, p. 174) *Let  $K$  be a shift-invariant  $\sigma$ -ideal of subsets of the real axis  $R$  such that*

$$(\forall Z)(Z \in K \rightarrow \lambda_*(Z) = 0),$$

*where  $\lambda_*$  denotes an inner measure defined by the linear Lebesgue measure  $\lambda$ . Then the functional  $\mu$  defined by*

$$\mu((X \cup Z') \cup Z'') = \lambda(X),$$

*where  $X$  is Lebesgue measurable subset of  $R$  and  $Z'$  and  $Z''$  are elements of the  $\sigma$ -ideal  $K$ , is a shift-invariant extension of the Lebesgue measure  $\lambda$ .*

For the proof of Lemma 2.1, see, e.g., [5], [7].

**Lemma 2.2.** ([5], Lemma 4, p. 164) *There exists a family  $(X_i)_{i \in [0,1]}$  of subsets of the real axis  $R$  such that:*

- 1)  $(\forall i)(\forall i')(i \in [0, 1] \ \& \ i' \in [0, 1] \ \& \ i \neq i' \rightarrow X_i \cap X_{i'} = \emptyset)$ ;
- 2)  $(\forall i)(\forall F)(i \in [0, 1] \ \& \ (F \text{ is a closed subset of the real axis } R \text{ with } \lambda(F) > 0) \rightarrow \text{card}(X_i \cap F) = c)$ ;
- 3)  $(\forall I')(\forall g)(I' \subseteq [0, 1] \ \& \ g \in R \rightarrow \text{card}(g + (\bigcup_{i \in I'} X_i) \triangle (\bigcup_{i \in I'} X_i)) < c)$ .

**Lemma 2.3.** *There exists a family  $(\mu_t)_{t \in [0,1]}$  of measures defined on some shift-invariant  $\sigma$ -algebra  $S(R)$  of subsets of the real axis  $R$  such that:*

- 1)  $(\forall t)(t \in [0, 1] \rightarrow \text{the measure } \mu_t \text{ is a shift-invariant extension of the linear Lebesgue measure } \lambda)$ ;
- 2)  $(\forall t)(\forall t')(t \in [0, 1] \ \& \ t' \in [0, 1] \ \& \ t \neq t' \rightarrow \mu_t \text{ and } \mu_{t'} \text{ are orthogonal}^2 \text{ measures. Moreover, } \mu_t(R \setminus X_t) = 0 \text{ for each } t \in [0, 1], \text{ where } (X_t)_{t \in [0,1]} \text{ comes from Lemma 2.2.})$ .

PROOF. For arbitrary  $t \in [0, 1]$ , we denote by  $K_t$  an shift-invariant  $\sigma$ -ideal generated by the set  $R \setminus X_t$ . Then it is easy to verify that the  $\sigma$ -ideal  $K_t$  satisfies all conditions of Lemma 2.1. Let us denote by  $\bar{\mu}_t$  the shift-invariant

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<sup>2</sup>  $\mu_t$  and  $\mu_{t'}$  are called orthogonal if there exists  $X \in S(R)$  such that  $\mu_t(X) = 0$  and  $\mu_{t'}(R \setminus X) = 0$ .

extension of the Lebesgue measure  $\lambda$  produced by the  $\sigma$ -ideal  $K_t$ . We obtain the family  $(\bar{\mu}_t)_{t \in [0,1]}$  of shift-invariant extensions of the Lebesgue measure  $\lambda$ .

Denote by  $S(R)$  the shift-invariant  $\sigma$ -algebra of subsets of the real axis  $R$ , generated by the union

$$F(R) \cup L(R) \cup \{X_t : t \in [0, 1]\},$$

where

$$F(R) = \{X : X \subseteq R \text{ \& card}(X) < c\}$$

and  $L(R)$  denotes the class of all Lebesgue measurable subsets of the real axis  $R$ .

Also, assume that

$$(\forall t)(t \in [0, 1] \rightarrow \mu_t = \bar{\mu}_t|_{S(R)}).$$

If we consider the family of shift-invariant measures  $(\mu_t)_{t \in [0,1]}$ , we can easily conclude that this family satisfies all conditions of Lemma 2.3.  $\square$

*Remark 2.4.* Let consider the family  $(\mu_t)_{t \in [0,1]}$  of shift-invariant extensions of the measure  $\lambda$  which comes from Lemma 2.3. Let denote by  $\lambda_t$  the restriction of the measure  $\mu_t$  to the class

$$S[0, 1] := \{Y \cap [0, 1] : Y \in S(R)\},$$

where  $S(R)$  comes from Lemma 2.3. It is obvious that for each  $t \in [0, 1]$ , the measure  $\lambda_t$  is concentrated on the set  $C_t = X_t \cap [0, 1]$  provided that  $\lambda_t([0, 1] \setminus C_t) = 0$ .

The next proposition is useful for our further consideration.

**Lemma 2.5.** (*Kolmogorov Strong Law of Large Numbers, [19], Theorem 3, p.379*) Let  $(\Omega, \mathcal{S}, P)$  be a probability space and  $(\xi_k)_{k \in \mathbf{N}}$  be a sequence of independent equally distributed random variables for which mathematical expectation  $m$  of  $\xi_1$  is finite. Then the following condition

$$P(\{\omega : \omega \in \Omega \text{ \& } \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \xi_k(\omega)}{n} = m\}) = 1$$

holds.

### 3 Uniformly distribution for invariant extensions of the Lebesgue measure defined by Remark 2.4

Let consider the family of probability measures  $(\lambda_t)_{t \in [0,1]}$  and the family  $(C_t)_{t \in [0,1]}$  of subsets of  $[0,1]$  which come from Remark 2.4.

**Lemma 3.1.** *For  $t \in [0,1]$ , we denote by  $\mathbf{L}([0,1], \lambda_t)$  the class of  $\lambda_t$ -integrable functions. Then for  $f \in \mathbf{L}([0,1], \lambda_t)$ , we have*

$$\lambda_t^\infty(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in [0,1]^\infty \text{ \& } \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n f(x_k)}{n} = \int_{[0,1]} f(x) d\lambda_t(x)\}) = 1.$$

PROOF. For fixed  $t \in [0,1]$ , we set

$$(\Omega, S, P) = (C_t^\infty, F(C_t)^\infty, \nu_t^\infty),$$

where

i)  $F(C_t) = \{C_t \cap Y : Y \in S[0,1]\}$ , where  $S[0,1]$  comes from Remark 2.4.

ii)  $\nu_t = \lambda_t|_{F(C_t)}$ , where  $\lambda_t|_{F(C_t)}$  denotes restriction of the measure  $\lambda_t$  to the sigma algebra  $F(C_t)$ .

For  $k \in N$  and  $(x_k)_{k \in N} \in C_t^\infty$  we put  $\xi_k((x_i)_{i \in N}) = f(x_k)$ . Then all conditions of Lemma 2.5 are satisfied which implies that

$$\nu_t^\infty(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in C_t^\infty \text{ \& } \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \xi_k((x_i)_{i \in N})}{n} = \int_{C_t^\infty} \xi_1((x_i)_{i \in N}) d\nu_t^\infty((x_i)_{i \in N})\}) = 1,$$

equivalently,

$$\nu_t^\infty(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in C_t^\infty \text{ \& } \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n f(x_k)}{n} = \int_{C_t} f(x) d\nu_t(x)\}) = 1.$$

The latter relation implies

$$\lambda_t^\infty(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in [0,1]^\infty \text{ \& } \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n f(x_k)}{n} = \int_{[0,1]} f(x) d\lambda_t(x)\}) \geq \nu_t^\infty(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in C_t^\infty \text{ \& } \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n f(x_k)}{n} = \int_{C_t} f(x) d\nu_t(x)\}) = 1.$$

□

**Definition 3.2.** A sequence of real numbers  $(x_k)_{k \in \mathbb{N}} \in [0, 1]^\infty$  is said to be  $\lambda$ -uniformly distributed sequence (abbreviated  $\lambda$ -u.d.s.) if for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d])}{n} = d - c. \quad (3.1)$$

We denote by  $S$  the set of all real valued sequences from  $[0, 1]^\infty$  which are  $\lambda$ -u.d.s. It is well known that  $(\{\alpha n\})_{n \in \mathbb{N}} \in S$  for each irrational number  $\alpha$ , where  $\{\cdot\}$  denotes the fractional part of the real number (cf. [11], Exercise 1.12, p. 16).

**Definition 3.3.** A sequence of real numbers  $(x_k)_{k \in \mathbb{N}} \in R^\infty$  is said to be uniformly distributed module 1 if the sequence it's fractional parts  $(\{x_k\})_{k \in \mathbb{N}}$  is  $\lambda$ -u.d.s.

*Remark 3.4.* It is obvious that  $(x_k)_{k \in \mathbb{N}} \in (0, 1)^\infty$  is uniformly distributed module 1 if and only if  $(x_k)_{k \in \mathbb{N}}$  is  $\lambda$ -u.d.s.

**Definition 3.5.** A sequence of real numbers  $(x_k)_{k \in \mathbb{N}} \in [0, 1]^\infty$  is said to be  $\lambda_t$ -uniformly distributed sequence (abbreviated  $\lambda_t$ -u.d.s.) if for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_t)}{n} = d - c. \quad (3.2)$$

We denote by  $S_t$  the set of all real valued sequences from  $[0, 1]^\infty$  which are  $\lambda_t$ -u.d.s.

In order to construct  $\lambda_t$ -u.d.s. for each  $t \in [0, 1]$ , we need the following lemma.

**Lemma 3.6.** ([11], THEOREM 1.2, p. 3) *If the sequence  $(x_n)_{n \in \mathbb{N}}$  is u.d. mod 1, and if  $(y_n)_{n \in \mathbb{N}}$  is a sequence with the property  $\lim_{n \rightarrow \infty} (x_n - y_n) = \alpha$  for some real constant  $\alpha$ , then  $(y_n)_{n \in \mathbb{N}}$  is u.d. mod 1.*

**Theorem 3.7.** *For each  $t \in [0, 1]$ , there exists  $\lambda_t$ -u.d.s.*

PROOF. Let consider a sequence  $(x_n)_{n \in \mathbb{N}} \in (0, 1)^\infty$  which is  $\lambda$ -u.d.s. For each  $n \in \mathbb{N}$ , we choose such an element  $y_n$  from the set  $C_t \cap (0, x_n)$  that  $|x_n - y_n| < \frac{1}{n}$ . This we can do because  $C_t$  is everywhere dense in  $(0, 1)$ . Now it is obvious that  $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ . By Lemma 3.6 we deduce that  $(y_n)_{n \in \mathbb{N}}$  is  $\lambda$ -u.d.s. Let us show that  $(y_n)_{n \in \mathbb{N}}$  is  $\lambda_t$ -u.d.s. Indeed, since  $y_k \in C_t$  for each  $k \in \mathbb{N}$  and  $(y_n)_{n \in \mathbb{N}}$  is  $\lambda$ -u.d.s., for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{y_k : 1 \leq k \leq n\} \cap [c, d] \cap C_t)}{n} =$$

$$\lim_{n \rightarrow \infty} \frac{\#(\{y_k : 1 \leq k \leq n\} \cap [c, d])}{n} = d - c. \quad (3.3)$$

□

**Theorem 3.8.** *For each  $t \in [0, 1]$ ,  $\lambda_t$ -u.d.s. is  $\lambda$ -u.d.s..*

PROOF. Let  $(x_k)_{k \in N}$  be  $\lambda_t$ -u.d.s. On the one hand, for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d])}{n} &\geq \\ \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_t)}{n} &= d - c. \end{aligned} \quad (3.4)$$

Since  $(x_k)_{k \in N}$  is  $\lambda_t$ -u.d.s., we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [0, 1] \cap C_t)}{n} = 1. \quad (3.5)$$

It is obvious that

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [0, 1])}{n} = 1. \quad (3.6)$$

The last two conditions implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap ([0, 1] \setminus C_t))}{n} &= \\ \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [0, 1])}{n} - & \\ \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap C_t)}{n} &= 1 - 1 = 0. \end{aligned} \quad (3.7)$$

The last relation implies that for each  $c, d$  with  $0 \leq c < d \leq 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap ([0, 1] \setminus C_t))}{n} &\leq \\ \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap ([0, 1] \setminus C_t))}{n} &= 0. \end{aligned} \quad (3.8)$$

Finally, for each  $c, d$  with  $0 \leq c < d \leq 1$  we get

$$\overline{\lim}_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d])}{n} \leq$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_t)}{n} + \\
& \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap ([0, 1] \setminus C_t))}{n} = \\
& (d - c) + 0 = d - c.
\end{aligned} \tag{3.9}$$

This ends the proof of theorem.  $\square$

*Remark 3.9.* Note that the converse to the result of Theorem 3.8 is not valid. Indeed, for fixed  $t \in [0, 1]$ , let  $(y_n)_{n \in N}$  be  $\lambda_t$ -u.d.s. which comes from Theorem 3.7. By Theorem 3.8,  $(y_n)_{n \in N}$  is  $\lambda$ -u.d.s. Let us show that  $(y_n)_{n \in N}$  is not  $\lambda_s$ -u.d.s. for each  $s \in [0, 1] \setminus \{t\}$ . Indeed, since  $y_k \in C_t$  for each  $k \in N$ , we deduce that  $y_k \notin C_s$  for each  $s \in [0, 1] \setminus \{t\}$ . The latter relation implies that for each  $s \in [0, 1] \setminus \{t\}$  and for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{y_k : 1 \leq k \leq n\} \cap [c, d] \cap C_s)}{n} = 0 < d - c. \tag{3.10}$$

*Remark 3.10.* For each  $\lambda$ -u.d.s.  $(y_n)_{n \in N}$  there exists a countable subset  $T \subset [0, 1]$  such that  $(y_n)_{n \in N}$  is not  $\lambda_t$ -u.d.s for each  $t \in [0, 1] \setminus T$ . Indeed, since  $\{C_t : t \in [0, 1]\}$  is the partition of the  $[0, 1]$ , for each  $k \in N$  there exists a unique  $t_k \in [0, 1]$  such that  $y_k \in C_{t_k}$ . Now we can put  $T = \cup_{k \in N} \{t_k\}$ .

**Theorem 3.11.** *There exists  $\lambda$ -u.d.s which is not  $\lambda_t$ -u.d.s. for each  $t \in [0, 1]$ .*

PROOF. Let consider a sequence  $(x_n)_{n \in N} \in (0, 1)^\infty$  which is  $\lambda$ -u.d.s. Since  $\{C_t : t \in [0, 1]\}$  is the partition of the  $[0, 1]$ , for each  $k \in N$  there exists a unique  $t_k \in [0, 1]$  such that  $y_k \in C_{t_k}$ . Now we can put  $T = \cup_{k \in N} \{t_k\}$ . Let  $S_0 = \{s_1, s_2, \dots\}$  be a countable subset of the set  $[0, 1] \setminus T$ . For each  $n \in N$ , we choose such element  $y_n$  from the set  $C_{s_n} \cap (0, x_n)$  that  $|x_n - y_n| < \frac{1}{n}$ . This we can do because  $C_t$  is everywhere dense in  $(0, 1)$  for each  $t \in [0, 1]$ . Now it is obvious that  $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ . By Lemma 3.6 we deduce that  $(y_n)_{n \in N}$  is  $\lambda$ -u.d.s. Let us show that  $(y_n)_{n \in N}$  is not  $\lambda_t$ -u.d.s. for each  $t \in [0, 1]$ . This follows from the fact that  $\text{card}(\{y_n : n \in N\} \cap C_t) \leq 1$  for each  $t \in [0, 1]$ . By this reason for each  $t \in [0, 1]$  and for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{\#(\{y_k : 1 \leq k \leq n\} \cap [c, d] \cap C_t)}{n} \leq \\
& \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < d - c.
\end{aligned} \tag{3.11}$$

$\square$



**Theorem 3.12.**  $S_i \cap S_j = \emptyset$  for each different  $i, j \in [0, 1]$ .

PROOF. Assume the contrary and let  $(x_k)_{k \in \mathbf{N}} \in S_i \cap S_j$ . On the one hand, for each  $c, d$  with  $0 \leq c < d \leq 1$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_i)}{n} = d - c. \quad (3.12)$$

On the other hand, for same  $c, d$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_j)}{n} = d - c. \quad (3.13)$$

By Theorem 3.8 we know that  $(x_k)_{k \in \mathbf{N}}$  is  $\lambda$ -u.d.s. which implies that for same  $c, d$  we have

$$\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d])}{n} = d - c. \quad (3.14)$$

But (3.14) is not possible because  $C_i \cap C_j = \emptyset$  which implies

$$\begin{aligned} d - c &= \lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d])}{n} \geq \\ &\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_i)}{n} + \\ &\lim_{n \rightarrow \infty} \frac{\#(\{x_k : 1 \leq k \leq n\} \cap [c, d] \cap C_j)}{n} = \\ &(d - c) + (d - c) = 2(d - c). \end{aligned} \quad (3.15)$$

We get the contradiction and theorem is proved.  $\square$

We have the following version of Hlawka's theorem(cf. [25]) for  $\lambda_t$ -uniformly distributed sequences.

**Theorem 3.13.** For  $t \in [0, 1]$ , we have  $\lambda_t^\infty(S_t) = 1$ .

PROOF. Let  $(f_k)_{k \in \mathbf{N}}$  be a countable subclass of  $L([0, 1], \lambda_t)$  which defines a  $\lambda_t$ -uniform distribution on  $[0, 1]$ .<sup>3</sup> For  $k \in N$ , we set

$$B_k = \{(x_k)_{k \in \mathbf{N}} : (x_k)_{k \in \mathbf{N}} \in [0, 1]^\infty \text{ \& } \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_k(x_n) = \int_{[0,1]} f_k(x) d\lambda_t x\}.$$

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<sup>3</sup>We say that a family  $(f_k)_{k \in \mathbf{N}}$  of elements of  $L([0, 1], \lambda_t)$  defines a  $\lambda_t$ -uniform distribution on  $[0, 1]$ , if for each  $(x_n)_{n \in \mathbf{N}} \in [0, 1]^\infty$  the validity of the condition  $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_k(x_n) = \int_{[0,1]} f_k(x) d\lambda_t(x)$  for  $k \in N$  implies that  $(x_n)_{n \in \mathbf{N}}$  is  $\lambda_t$ -u.d.s. Indicator functions of sets  $[c, d] \cap C_t$  with rational  $c, d$  is an example of such a family.

By Lemma 3.1 we know that  $\lambda_t^\infty(B_k) = 1$  for  $k \in \mathbf{N}$ , which implies  $\lambda_t^\infty(\cap_{k \in \mathbf{N}} B_k) = 1$ . Hence

$$\lambda_t^\infty(\{(x_k)_{k \in \mathbf{N}} : (x_k)_{k \in \mathbf{N}} \in [0, 1]^\infty \text{ \& } (\forall k)(k \in \mathbf{N} \rightarrow \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_k(x_n) = \int_{[0,1]} f_k(x) d\lambda_t(x))\}) = 1.$$

The latter relation means that  $\lambda_t^\infty$ -almost every elements of  $[0, 1]^\infty$  is  $\lambda_t$ -u.d.s., equivalently,  $\lambda_t^\infty(S_t) = 1$ . □

We have the following analogue of H.Weyl theorem ( cf. [22] ) for  $\lambda_t$ -uniformly distributed sequences.

**Theorem 3.14.** *For  $t \in [0, 1]$ , we put  $C_t[0, 1] = \{\tilde{h}(x) = h(x) \times \chi_{C_t}(x) : h \in C[0, 1]\}$ . Then the sequence  $(x_n)_{n \in \mathbf{N}}$  is  $\lambda_t$ -u.d.s. if and only if the following condition*

$$\lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N \tilde{h}(x_n)}{N} = \int_{[0,1]} \tilde{h}(x) d\lambda_t(x) \quad (3.16)$$

holds for each  $\tilde{h} \in C_t[0, 1]$ .

PROOF. Let  $(x_n)_{n \in \mathbf{N}}$  be  $\lambda_t$ -u.d.s. and let  $f(x) = \sum_{i=1}^{k-1} d_i \chi_{[a_i, a_{i+1}] \cap C_t}(x)$  be a spatial step function on  $[0, 1]$ , where  $0 = a_0 < a_1 < \dots < a_k = 1$ . Then it follows from (3.2) that for every such  $f$  equation (3.16) holds. We assume now that  $\tilde{f} \in C_t[0, 1]$ . Given any  $\epsilon > 0$ , there exist, by the definition of the Riemann integral, two step functions,  $f_1$  and  $f_2$  say, such that  $f_1(x) < f(x) < f_2(x)$  for all  $x \in [0, 1]$  and

$$\int_{[0,1]} (f_2(x) - f_1(x)) d\lambda(x) < \epsilon.$$

Then it is obvious that  $f_1(x) \chi_{C_t}(x) < \tilde{f}(x) < f_2(x) \chi_{C_t}(x)$  for all  $x \in [0, 1]$  and

$$\int_{[0,1]} (f_2(x) \chi_{C_t}(x) - f_1(x) \chi_{C_t}(x)) d\lambda_t(x) = \int_{[0,1]} (f_2(x) - f_1(x)) d\lambda(x) < \epsilon.$$

Then we have the following chain of inequalities:

$$\int_{[0,1]} \tilde{f}(x) d\lambda_t(x) - \epsilon \leq \int_{[0,1]} f_1(x) \chi_{C_t}(x) d\lambda_t(x) =$$

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N f_1(x_n) \chi_{C_t}(x_n)}{N} \leq \\
& \underline{\lim}_{N \rightarrow \infty} \frac{\sum_{n=1}^N f(x_n) \chi_{C_t}(x_n)}{N} = \\
& \underline{\lim}_{N \rightarrow \infty} \frac{\sum_{n=1}^N \tilde{f}(x_n)}{N} \leq \\
& \overline{\lim}_{N \rightarrow \infty} \frac{\sum_{n=1}^N \tilde{f}(x_n)}{N} = \\
& \overline{\lim}_{N \rightarrow \infty} \frac{\sum_{n=1}^N f(x_n) \chi_{C_t}(x_n)}{N} \leq \\
& \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N f_2(x_n) \chi_{C_t}(x_n)}{N} = \\
& \int_{[0,1]} f_2(x) \chi_{C_t}(x) d\lambda_t(x) \leq \\
& \int_{[0,1]} \tilde{f}(x) d\lambda_t(x) + \epsilon. \tag{3.18}
\end{aligned}$$

So that in the case of a function  $\tilde{f}$  the relation (3.16) holds.

Conversely, let a sequence  $(x_n)_{n \in \mathbb{N}}$  be given, and suppose that (3.16) holds for every  $\tilde{f} \in C_t[0, 1]$ . Let  $[a, b]$  be an arbitrary subinterval of  $[0, 1]$ . Given any  $\epsilon > 0$ , there exist two continuous functions,  $g_1$  and  $g_2$  say, such that  $g_l(x) < \chi_{[a,b]}(x) < g_2(x)$  for  $x \in [0, 1]$  and at the same time  $\int_{[0,1]} (g_2(x) - g_l(x)) d\lambda x < \epsilon$ . Note that at the same time we have  $g_l(x) \chi_{C_t}(x) < \chi_{[a,b] \cap C_t}(x) < g_2(x) \chi_{C_t}(x)$  for  $x \in [0, 1]$  and  $\int_{[0,1]} (\tilde{g}_2(x) - \tilde{g}_l(x)) d\lambda_t x < \epsilon$ .

Then we get

$$\begin{aligned}
b - a - \epsilon & < \int_{[0,1]} g_2(x) d\lambda x - \epsilon < \int_{[0,1]} g_1(x) d\lambda x = \\
& \int_{[0,1]} \tilde{g}_1(x) d\lambda_t x = \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N \tilde{g}_1(x_n)}{N} \leq \\
& \underline{\lim}_{N \rightarrow \infty} \frac{\#(\{x_1, \dots, x_N\} \cap [a, b] \cap C_t)}{N} \leq \\
& \overline{\lim}_{N \rightarrow \infty} \frac{\#(\{x_1, \dots, x_N\} \cap [a, b] \cap C_t)}{N} \leq \\
& \lim_{N \rightarrow \infty} \frac{\sum_{n=1}^N \tilde{g}_2(x_n)}{N} =
\end{aligned}$$

$$\begin{aligned}
\int_{[0,1]} \tilde{g}_2(x) d\lambda_t x &= \int_{[0,1]} g_2(x) d\lambda x \\
&\leq \int_{[0,1]} g_1(x) d\lambda x + \epsilon \leq \\
\int_{[0,1]} \chi_{[a,b]}(x) d\lambda x + \epsilon &= b - a + \epsilon.
\end{aligned} \tag{3.19}$$

Since  $\epsilon$  is arbitrarily small, we have (3.2).  $\square$

## 4 Historical Background for invariant extensions of the Haar measure

### 4.1 On Waclaw Sierpiniski problem

By Vitali's celebrate theorem about existence of the linear Lebesgue non-measurable subset has been shown that the domain of the Lebesgue measure in  $R$  differs from the power set of the real axis  $R$ . In this context the following question was naturally appeared:

*"How far can we extend Lebesgue measure and what properties can such an extension preserve?"*

In 1935 E. Marczewski, applied Sierpiński construction of an almost invariant set  $A$ , obtained a proper invariant extension of the Lebesgue measure in which the extended  $\sigma$ -algebra had contained new sets of positive finite measure. In connection with this result, Waclaw Sierpiniski in 1936 posed the following

**Problem (Waclaw Sierpiński)** *Let  $D_n$  denotes the group of all isometrical transformations of the  $R^n$ . Does there exist any maximal  $D_n$ -invariant measure?*

The first result in this direction was obtained by Andrzej Hulanicki [27] as follows:

**Proposition** ( Andrzej Hulanicki (1962)) *If the continuum  $2^{\aleph_0}$  is not real valued measurable cardinal then there does not exist any maximal invariant extension of the Lebesgue measure.*

This result was also obtained independently by S. S. Pkhakadze [18] using similar methods.

In 1977 A. B. Kharazishvili got the same answer in the one-dimensional case without any set-theoretical assumption (see [4] ).

Finally, in 1982 Krzysztof Ciesielski and Andrzej Pelc generalized Kharazishvili's result to all  $n$ -dimensional Euclidean spaces (see [1]).

Following Solovay [20], if the system of axioms "ZFC & There exists inaccessible cardinal" is consistent then the systems of axioms "ZF & CD & Every set of reals is Lebesgue measurable" is also consistent. This result implies that the answer to Wacław Sierpiński's problem is affirmative.

Taking Solovay's result on the one hand, and Krzysztof Cieśliński and Andrzej Pelc (or Andrzej Hulanicki or Pkhakadze) result on the other hand, we deduce that the Wacław Sierpiński's question is not solvable within the theory ZF & CD.

#### 4.2 On Lebesgue measure's invariantly extension methods in ZFC

Now days there exists a reach methodology for a construction of invariant extensions of the Lebesgue measure in  $R^n$  as well the Haar measure in a locally compact Hausdorff topological group. Let us briefly consider main of them.

**Method I.** (Jankowka-Wiatr) Following [28], the first idea of extending the Lebesgue measure in  $R^n$  to a larger  $\sigma$ -algebra in such a way that it remains invariant under translations belongs to Jankowka-Wiatr who in 1928 observed that one can add new sets to the  $\sigma$ -ideal of sets of Lebesgue measure zero and still preserve the invariance of the extended measure. This method can be described as follows:

Let  $K$  be a shift-invariant  $\sigma$ -ideal in the  $n$ -dimensional Euclidean space  $R^n$  such that

$$(\forall Z)(Z \in K \rightarrow m_*(Z) = 0),$$

where  $m_*$  denotes the inner measure defined by  $n$ -dimensional Lebesgue measure  $m$ . Then the functional  $\overline{m}$  defined by

$$\overline{m}((X \cup Z') \setminus Z'') = m(X),$$

where  $X$  is a Borel subset of  $R^n$  and  $Z'$  and  $Z''$  are elements of the  $\sigma$ -ideal  $K$ , is an  $D_n$ -invariant extension of the Lebesgue measure  $m$ .

**Method II.** (E. Szpilrajn (E. Marczewski)) By using Sierpiński's decomposition  $\{A, B\}$  of the  $R^2$ , E. Szpilrajn noted that the following two conditions

- (i)  $\text{card}(A \text{ triandle}(x + A)) < c$ ,  $\text{card}(A \Delta (x + A)) < c$  for each  $x \in R^2$ ;
- (ii)  $\text{card}(A \cap F) = \text{card}(B \cap F) = 2^{\aleph_0}$  for each closed set  $F \subseteq R^2$  with  $m(F) > 0$ .

holds true.

Further, he constructed a proper shift-invariant extension  $\overline{m}$  of the Lebesgue measure  $m$  in  $R^2$  as follows

$$\overline{m}((A \cap X) \cup (B \cap Y)) = 1/2(m(X) + m(Y))$$

for  $X, Y \in \text{dom}(m)$ .

**Method III.** ( Oxtoby and Kakutani ) Some methods of combinatorial set theory have lately been successfully used in measure extension problem. Among them, special mention should be made of the method of constructing a maximal (in the sense of cardinality) family of independent families of sets in arbitrary infinite base spaces. The question of the existence of a maximal (in the sense of cardinality)  $\aleph_0$ -independent <sup>4</sup>family of subsets of an uncountable set  $E$  was considered by A. Tarski. He proved that this cardinality is equal to  $2^{\text{card}(E)}$ .

This result found an interesting application in general topology. For example, it was proved that in an arbitrary infinite space  $E$  the cardinality of the class of all ultrafilters is equal to  $2^{2^{\text{card}(E)}}$  (see, e.g., [12]).

The combinatorial question of the existence of a maximal (in the sense of cardinality) strict  $\aleph_0$ -independent family of subsets of a set  $E$  with cardinality of the continuum also was investigated and was proved that this cardinality is equal to  $2^c$ .

This combinatorial result found an interesting application in the Lebesgue measure theory. For example, Kakutani and Oxtoby [3] firstly constructed a family  $\mathcal{A}$  of almost invariant subsets of the circle in such a way that

$$\bigcap_{n=1}^{\infty} A_n^{\epsilon_n}$$

has outer measure 1 for an arbitrary sequence  $\{A_n\}$  of sets from  $\mathcal{A}$  and arbitrary sequence  $\{\epsilon_n\}, \epsilon_n = 0, 1$ . The putting  $\overline{m}(A) = 1/2$  for  $A$  in  $\mathcal{A}$  they obtained an extension of the Lebesgue measure on the circle to an invariant measure  $\overline{m}$  such that  $L_2(\overline{m})$  has the Hilbert space dimensional equal to  $2^c$ .

Using the same combinatorial result, A.B. Kharazishvili constructed a maximal (in the sense of cardinality) family of orthogonal elementary  $D_n$ -invariant extensions of the Lebesgue measure (see [5]).

The combinatorial question of the existence of a maximal (in the sense of cardinality) strict  $\aleph_0$ -independent family of subsets of a set  $E$  with  $\text{card}(E^{\aleph_0}) = \text{card}(E)$  was investigated in [15] and it was shown that this cardinality is equal to  $2^{\text{card}(E)}$ . Using this result, G.Pantsulaia [13] extended Kakutani and Oxtoby [3] method for a construction of a maximal (in the sense of cardinality) family of orthogonal elementary  $H$ -invariant extensions of the Haar measure defined in a locally compact  $\sigma$ -compact topological group with  $\text{card}(H^{\aleph_0}) = \text{card}(H)$ .

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<sup>4</sup>We say that a family  $(X_i)_{i \in I}$  of subsets of the set  $E$  is  $\aleph_0$ -independent if the condition  $(\forall J)(J \subset I \ \& \ \text{card}(J) < \aleph_0 \rightarrow \bigcap_{i \in J} \overline{X}_i \neq \emptyset)$  holds, where  $(\forall i)(i \in I \rightarrow (\overline{X}_i = X_i) \vee (\overline{X}_i = (E \setminus X_i)))$ . If in addition, this condition holds true for each  $J$  with  $\text{card}(J) \leq \aleph_0$ , then  $(X_i)_{i \in I}$  is called strict  $\aleph_0$ -independent.

**Method IV.**( Kodaira and Kakutani method) Kodaira and Kakutani [10] invented the following method of extended the Lebesgue measure on the circle to an invariant measure as follows:

Let produce a *character*  $\pi$  of the circle, i.e. a homomorphism  $\pi : T \rightarrow T$  in such a way that the outer Lebesgue measure of its graph  $D_\pi$  is equal to 1 in  $T \times T$ . Then the extended  $\sigma$ -algebra  $\overline{B}$  consists of sets  $A_M = \{x : (x, \pi(x)) \in M\}$ , where  $M$  is Lebesgue measurable set in  $T \times T$  and the extended measure  $\overline{m}$  is  $\overline{m}(A_M) = (m \times m)(M)$ . Note that the discontinuous character  $\pi$  becomes  $\overline{B}$ -measurable. It has been noticed later in [26] that one can produce  $2^c$  such characters so that they all become measurable and  $L^2(\overline{m})$  is of Hilbert space dimension  $2^c$ .

This method have been modified for  $n$ -dimensional Euclidean space in [15](Received 7. November 1993) for a construction of the invariant extension  $\mu$  of the  $n$ -dimensional Lebesgue measure such that there exists a  $\mu$ -measurable set with only one density point. This result answered positively to a certain question stated by A.B. Kharazishvili (cf. [5], Problem 9, p. 200). Knowing this result, A.B. Kharazishvili considered similar but originally modified method and extended previous result in [6]( Received 15. March 1994) as follows: *there exists an invariant extension  $\mu$  of the classical Lebesgue measure such that  $\mu$  has the uniqueness property and there exists a  $\mu$ -measurable set with only one density point..*

**Method  $\star$ .** More lately, Kodaira and Kakutani method have been modified for an uncountable locally compact  $\sigma$ -compact topological group  $H$  with  $\text{card}(H^{\aleph_0}) = \text{card}(H)$  in [16] as follows: Let  $E$  be a set with  $2 \leq \text{card}(E) \leq \text{card}(H)$  and let  $\mu$  be a probability measure in  $E$  such that each  $X \in \text{dom}(\mu)$  for which  $\text{card}(X) < \text{card}(E)$ . Let produce a *function*  $f : H \rightarrow E$  in such a way that the following two conditions

- 1)  $(\forall e)(\forall F)(e \in E \ \& \ (F \text{ is a closed subset of the } H \text{ with } m(F) > 0) \rightarrow \text{card} f^{-1}(e) \cap F = \text{card}(E))$ ;
- 2)  $(\forall E')(\forall g)(E' \subseteq E \ \& \ g \in H \rightarrow \text{card}(g(\bigcup_{e \in E'} f^{-1}(e)) \Delta (\bigcup_{e \in E'} f^{-1}(e))) < \text{card}(H))$ .

holds true. Then the extended  $\sigma$ -algebra  $\overline{B}$  consists of sets  $A_M = \{x : (x, f(x)) \in M\}$ , where  $M \in \text{dom}(m) \times \text{dom}(\mu)$ . Then the extended measure  $\overline{m}_\mu$  is defined by  $\overline{m}_\mu(A_M) = (m \times \mu)(M)$ . Note that  $\overline{m}_\mu$  is a non-elementary invariant extension of the measure  $m$  iff the measure  $\mu$  is diffused. It has been noticed that one can produce  $2^{\text{card}(H)}$  such functions so that they all become measurable and  $L^2(\overline{m}_\mu)$  is of Hilbert space dimension  $2^{\text{card}(H)}$ .

Note that when  $\text{card}(E) = 2$ ,  $\mu$  is a normalized counting measure in  $E$  and  $f : R^2 \rightarrow E$  is defined by  $f(x) = \chi_A(x)$ , then Method  $\star$  gives Marczewski method.

When  $H = E = T$ ,  $\mu = m$  and  $f = \pi$  is a character, then Method  $\star$  gives Kodaira and Kakutani method.

Now let us discuss whether Method  $\star$  gives Oxtoby and Kakutani method. In this context we need some auxiliary facts.

**Lemma 4.1.** ( [17], **Theorem 11.1** , p. 158 ) *If an infinite set  $E$  satisfies the condition*

$$\text{Card}(E^{\aleph_0}) = \text{card}(E),$$

*then there exists a maximal (in the sense of cardinality) strictly  $\aleph_0$ -independent family  $(A_i)_{i \in I}$  of subsets of the space  $E$ , such that*

$$\text{card}(I) = 2^{\text{card}(E)}.$$

**Lemma 4.2.** ( [17], **Lemma 11.2** , p. 163 ) *Let  $H$  be an arbitrary locally compact  $\sigma$ -compact topological group,  $\lambda$  be the Haar measure defined on the group  $H$  and let  $\alpha$  be an arbitrary cardinal number such that:*

$$\alpha \leq \text{card}(H).$$

*Then there exists a family  $(X_i)_{i \in I}$  of subsets of the set  $H$  such that:*

- 1)  $\text{Card}(I) = \alpha$ ;
- 2)  $(\forall i)(\forall i')(i \in I \ \& \ i' \in I \ \& \ i \neq i' \rightarrow X_i \cap X_{i'} = \emptyset)$ ;
- 3)  $(\forall i)(\forall F)(i \in I \ \& \ (F \text{ is a closed subset of the space } H \text{ with } \lambda(F) > 0) \rightarrow \text{card}(X_i \cap F) = \text{card}(H))$ ;
- 4)  $(\forall I')(\forall g)(I' \subseteq I \ \& \ g \in H \rightarrow \text{card}(g(\bigcup_{i \in I'} X_i) \Delta (\bigcup_{i \in I'} X_i)) < \text{card}(H))$ .

**Lemma 4.3.** ( [17], **Lemma 11.5** , p. 169 ) *Let  $E$  be an uncountable base space with  $\text{card}(E^{\aleph_0}) = \text{card}(E)$ . Then there exists a non-atomic probability measure  $\mathcal{P}$  such that the following conditions hold:*

- a)  $(\forall X)(X \subseteq E \ \& \ \text{card}(X) < \text{card}(E) \rightarrow \mathcal{P}(X) = 0)$ ;
- b) *the topological weight  $a(\mathcal{P})$  of the metric space  $(\text{dom}(\mathcal{P}), \rho_{\mathcal{P}})$  associated with measure  $\mathcal{P}$  is maximal, in particular, is equal to  $2^{\text{card}(E)}$ .*

Now put  $H = G$  and  $E = \{0, 1\}^N$ . For  $g \in G$ , we put  $h(g) = (g, i)$ , where  $i$  is a unique index for which  $g \in X_i$  (cf. Lemma 4.2 for  $\alpha = \text{card}(H) = 2^{\aleph_0}$ ). We set  $\overline{A}_i = \bigcup_{e \in X_i} f^{-1}(e)$  for  $i \in I$ . Let  $\mathcal{P}$  comes from 4.3. Now if we consider the invariant extension  $\overline{m}_{\mathcal{P}}$  of the measure  $m$ , we observe that  $\overline{m}_{\mathcal{P}}(\overline{A}_s) = 1/2$  for each  $\overline{A}_s \in \mathcal{A} := \{\overline{A}_i : i \in I\}$ . Since  $\mathcal{A}$  is the family of strictly  $\aleph_0$ -independent almost invariant subsets of  $G$ , we claim that Method  $\star$  gives just above described Oxtoby and Kakutani method.

**Method V.** ( Kharazishvili ). This approach, as usual, can be used for uncountable commutative groups and is based on purely algebraic properties



those groups, which are not assumed to be endowed with any topology but only are equipped with a nonzero  $\sigma$ -finite invariant measures. Here essentially is used Kulikov's well known theorem about covering of any commutative group by increasing (in the sense of inclusion) countable sequence of subgroups of  $G$  which are direct sum of cyclic groups (finite or infinite) (see, for example [8], [9]).

**Definition 4.4(Kharazishvili)** Let  $E$  be a base space,  $G$  be a group of transformations of  $E$  and let  $X$  be a subset of the space  $E$ .  $X$  is called a  $G$ -absolutely negligible set if for any  $G$ -invariant  $\sigma$ -finite measure  $\mu$ , there exists its  $G$ -invariant extension  $\bar{\mu}$  such that  $X \in \text{dom}(\bar{\mu})$  and  $\bar{\mu}(X) = 0$ .

A geometrical characterization of absolutely negligible subsets, due to A.B. Kharazishvili, is presented in the next proposition.

**Theorem 4.5** *Let  $E$  be a base space,  $G$  be a group of transformations of  $E$  containing some uncountable subgroup acting freely in  $E$ , and  $X$  be an arbitrary subset of the space  $E$ . Then the following two conditions are equivalent:*

- 1)  $X$  is a  $G$ -absolutely negligible subset of the space  $E$ ;
- 2) for an arbitrary countable  $G$ -configuration<sup>5</sup>  $X'$  of the set  $X$ , there exists a countable sequence  $(h_k)_{k \in \mathbb{N}}$  of elements of  $G$

$$\bigcap_{k \in \mathbb{N}} h_k(X') = \emptyset.$$

It is of interest that the class of all countable  $G$ -configurations of the fixed  $G$ -absolutely negligible subset constitutes a  $G$ -invariant  $\sigma$ -ideal such that the inner measure of each element of this class is zero with respect to any  $\sigma$ -finite  $G$ -invariant measure in  $E$ . Hence, by using the natural modification of the Method I one can obtain  $G$ -invariant extension of an arbitrary  $\sigma$ -finite  $G$ -invariant measure in  $E$ .

In 1977 A. B. Kharazishvili constructed the partition of the real axis  $R$  in to the countable family of  $D_1$ -absolutely negligible sets and got the negative answer to the question of Waclaw Sierpiński in the one-dimensional case without any set-theoretical assumption (see [4]).

Finally, in 1982 Krzysztof Ciesielski and Andrzej Pelc generalized Kharazishvili's result to all  $n$ -dimensional Euclidean spaces, more precisely, they constructed the partition of the Euclidean space  $R^n$  in to the countable family of  $D_n$ -absolutely negligible sets and got the negative answer to the question of Waclaw Sierpiński in the  $n$ -dimensional case without any set-theoretical assumption (see [1]).

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<sup>5</sup>A subset  $X'$  of  $E$  is called a countable  $G$ -configuration of  $X$  if there is a countable family  $\{g_k : k \in \mathbb{N}\}$  of elements of  $G$  such that  $X' \subseteq \bigcup_{k \in \mathbb{N}} g_k(X)$ .

By using the method of absolutely negligible sets elaborated by A.Kharazishvili [5], P. Zakrzewski [29] answered positively to a question of Ciesielski [2] asking *whether an isometrically invariant  $\sigma$ -finite countably additive measure on  $R^n$  admits a strong countably additive isometrically invariant extension*. It is obvious that this question is generalization of the above mentioned Wacław Sierpiński problem.

## 5 Discussion

Let  $G$  be a compact Hausdorff topological group and  $\lambda$  be a Haar measure in  $H$ . By  $B(G)$  we mean the set of all bounded real-valued Borel-measurable functions on  $G$ . Under the norm  $\|f\| = \sup_{g \in G} |f(g)|$  for  $f \in B(G)$ , the set  $B(G)$  forms a Banach space, and even a Banach algebra if algebraic operations for functions are defined in the usual way. The subset  $R(G)$  of  $B(G)$  consisting of all real-valued continuous functions on  $G$  is then a Banach subalgebra of  $B(G)$ .

Following [11] (see Definition 1.1, p.171), The sequence  $(x_n)_{n \in \mathbb{N}}$  of elements in  $G$  is called  $\lambda$ -u.d. in  $G$  if

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n f(x_k)}{n} = \int_G f(x) d\lambda(x) \quad (5.1)$$

for all  $f \in R(G)$ .

Note that the theory of uniform distribution is well developed in compact Hausdorff topological groups (see, for example [11], Chapter 4) as well the theory of invariant extensions of Haar measures in the same groups (see Section 4). Here naturally arise a question asking *whether can be introduced the concept of uniform distribution for invariant extensions of the Haar measure in compact Hausdorff topological groups*. We wait that by using similar manner used in Section 3 and the methodology briefly described in Section 4, one can resolve this question.

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